LECTURE NOTES 2-4: THE PRECISE DEFINITION OF THE LIMIT

REVIEW: List our present strategies for determining $\lim_{x\to a} f(x)$, if it exists.

What are some of the weaknesses in these approaches?

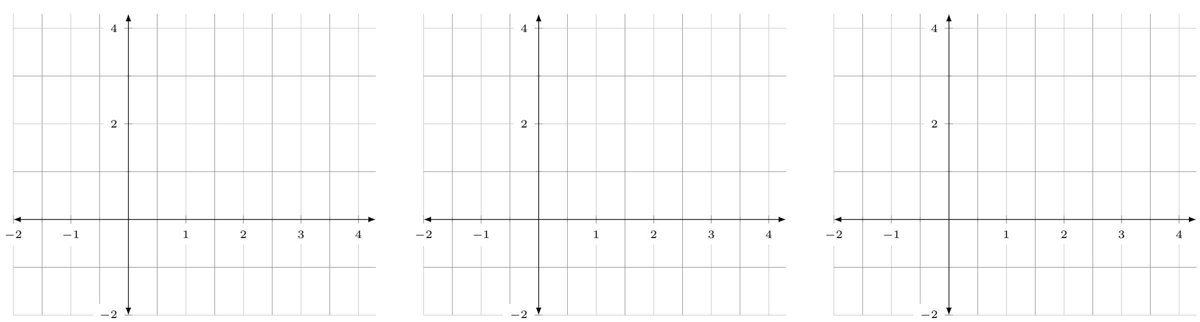
GOALS:

- Experience the precise (or *formal*) definition of the limit.
- Confirm student intuition that there is a lot going on when evaluating limits that may not be crystal clear!
- "Look under the hood" of the mathematics used in Calculus (& Differential Equations).

PRACTICE PROBLEMS:

1. Graph the region of the xy-plane satisfying each of the inequalities below.

$|x-3| < 1$ $\qquad$ $|x-3| < 1/2$ $\qquad$ $|y-1| < 1/3$

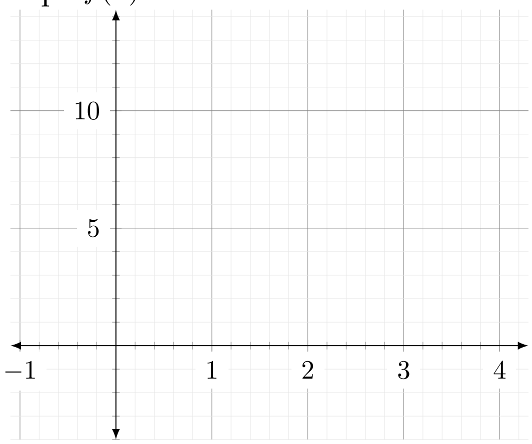


2. Graph the region of the xy-plane satisfying the inequality $|x-a| < c$.

3. Re-write the expression $|x-a| < c$ using *interval notation*.

4. Let $f(x) = 3x + 1$.

(a) Graph $f(x)$.



(b) If the domain of $f(x)$ is restricted to

$$|x - 2| < 1,$$

what would the range of $f(x)$ be? Sketch the intervals representing domain and range on the graph.

(c) If the domain of $f(x)$ is restricted to

$$|x - 2| < \frac{1}{5},$$

what would the range of $f(x)$ be? Sketch the intervals representing domain and range on the graph.

(d) **We are now going to switch our point of view so READ CAREFULLY!** Assume you must "hit a target" in the range. Specifically, assume you need the output of the function to lie in the region $|y-7| < 1$, how would you need to restrict your domain? Is your answer UNIQUE? Give you final answer in the form $|x-a| < c$, for some a and c.

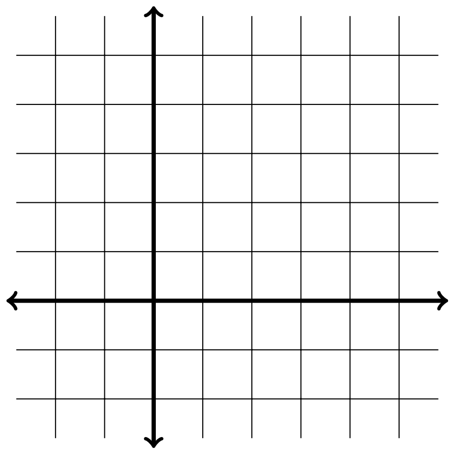
(e) Repeat the previous question but replace the target interval $|y - 7| < 1$ with $|y - 7| < \frac{1}{10}$.

(f) Repeat the previous question but replace the target interval with $|y - 7| < \epsilon$, where ϵ is some unknown but fixed positive number. (Note the symbol ϵ is called *epsilon* and in mathematics is very nearly always used to represent some really small positive number, 0.0000001, for example.)

THE PRECISE DEFINITION OF THE LIMIT:

We say $\lim_{x \to a} f(x) = L$, if for *every* number $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$



5. Use the precise definition of the limit to show $\lim_{x \to 2} 3x + 1 = \boxed{}$.

6. In words in English, write down how to apply the precise definition of the limit. Assume you are given a function, $f(x)$ and an x-value a and asked to find $\lim_{x \to a} f(x)$.